Distribution theoretic approach to fictitious domain method for Neumann problems(Nonlinear Mathematical Problems in Industry)

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数理解析研究所講究録
891 巻 1995 年 10-40

Distribution theoretic approach to fictitious domain method for Neumann problems

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Abstract

In this paper, some kind of fictitious domain method via singular perturbation is presented in order to solve boundary value problems for partial differential equations prescribed with Neumann boundary conditions.

The characteristic of this method is to reformulate the original problem into the one where Neumann data are represented as a well defined distribution supported by Γ. Γ is the boundary of a domain in which the original problem is defined. This definition of the distribution makes it inevitable to have a fictitious domain adjacent to Γ and outside of the domain, on which we have no governing equations.

Therefore, the regularization of the original problem from the outside is needed by using singularly perturbed equations.

This method is applied to the flow problems coupled with the unilateral boundary conditions of the Signorini type. Finally numerical solutions for them are obtained by means of finite difference approximation.

1. Introduction

We propose a kind of fictitious domain method which is effectively used to solve a boundary value problem with nonhomogeneous Neumann boundary condition. Nowadays there are a variety of methods which are commonly called fictitious domain method. As typical examples we can refer to matrix capacitance method[6],[33],[35], fictitious domain method via harmonic extensions[3],[17],[22], fictitious domain method via Lagrange multipliers and optimal control[4],[16] and fictitious domain method via singular perturbations [9],[18],[19],[20],[25],[30],[31]. All these make use of the fictitious domain in some way or other, but their underlying ideas and algorithm differ.

Our method belongs to the category of fictitious domain method via singular perturbation. This paper is divided into two parts. In the first part, we present our methodology for a simple boundary value problem defined in a bounded domain in R^2, while its generalization is straightforward. The second one is devoted to an application of the method to the variational inequality which is equivalent to the boundary value problem with unilateral boundary condition of the friction type [5],[7],[8],[15],[21],[27],[36],[37] for stationary Navier-Stokes equations. Physically speaking, this problem is to study phenomena that the fluid leaks and / or slips on some part of the boundary. As an example, we consider the flow in a rectangular duct with inlet-outlet boundaries and have obtained numerical solutions for this problem by means of finite difference approximation.

2. Fictitious domain method via singular perturbation.

2.1

To fix the idea, we begin with the boundary value problem (P) for scalar functions with the Neumann boundary condition. Ω_0 is assumed to be a bounded domain in R^2 with boundary Γ equipped with suitable regularity .

$$(P)\quad \begin{cases} (1) \quad -\nu \triangle u + u = f & \text{in } \Omega_0, \\ (2) \quad \dfrac{\partial u}{\partial n} = g & \text{on } \Gamma. \end{cases}$$

Here $f \in L^2(\Omega_0)$, $g \in H^{-1/2}(\Gamma)$; ν is a positive constant and n is the unit outward normal vector to Γ.

It is well known that (P) has a unique solution $u \in H^1(\Omega_0)$. For (P), we introduce three approximate problems parametrized with ε (> 0) defined in a wider domain B according to the general idea of the fictitious domain method via singular perturbation.
B includes $\overline{\Omega}_0$, and with $\Omega_1 = B \setminus \overline{\Omega}_0$, we can write $B = \Omega_0 \cup \Gamma \cup \Omega_1$.

This extension makes it possible to define a distribution supported by Γ, which represents Neumann data. We have no governing equations on Ω_1. Then let us regularize the original problem from the outside of Ω_0 by using singularly perturbed equations, which makes approximate (2).

The purpose of this section is to prove the equivalence relations among them and the convergence of an approximate solution into the solution of (P) as ε tends to 0. In order to make the argument about the proof be simpler, we assume that the boundary Γ is smooth and g is a smooth function defined on Γ. Then n and g have smooth extensions in B, which are denoted by $\tilde{n}$ and $\tilde{g}$ respectively.

One of three approximate problems is the following [26];

Find $u_0 \in H^1(\Omega_0)$ and $u_1 \in H^1(\Omega_1)$ such that

$(P)_\varepsilon$

(3) $-\nu \triangle u_0 + u_0 = f$ in Ω_0 (in $H^{-1}(\Omega_0)$),

(4) $-\varepsilon^{2\alpha} \triangle u_1 = 0$ in Ω_1 (in $H^{-1}(\Omega_1)$),

(5) $u_0 = u_1$ on Γ (in $H^{\frac{1}{2}}(\Gamma)$),

(6) $\nu \dfrac{\partial u_0}{\partial n} = \varepsilon^{2\alpha} \cdot \dfrac{\partial u_1}{\partial n} + g$ on Γ (in $H^{-\frac{1}{2}}(\Gamma)$),

(7) $u_1 = 0$ on ∂B (in $H^{\frac{1}{2}}(\partial B)$).

Here ε (> 0) is a small parameter, while α is a fixed positive constant.

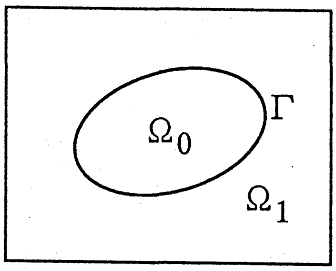

Figure 2.1

For $(P)_\varepsilon$, we have the following variational formulation ;

Find $u \in H^1_0(B)$ such that

$(P)'_\varepsilon$

(8) $\nu a_0(u, v) + \varepsilon^{2\alpha} \cdot a_1(u, v) + (u, v)_0 + \int_\Gamma g v d\Gamma$

$= (f, v)_0$, for any $v \in H^1_0(B)$.

Here $a_k(u, v) = \int_{\Omega_k} \nabla u \cdot \nabla v \, dx$ (k=0, 1) and $$(u,v)_0 = \int_{\Omega_0} uv\,dx.$$

Here $u = u_0$ (in Ω_0) and $u = u_1$ (in Ω_1).

Let χ be the characteristic function of Ω_1 in B. By means of χ, we define an extended flux as follows ;

$$(9) \qquad \sigma^e(u) = \nu \cdot (1-\chi) \cdot \nabla u_0 + \varepsilon^{2\alpha} \cdot \chi \cdot \nabla u_1 .$$

Then we state $(P)''_\varepsilon$ below, which is the distribution version of $(P)_\varepsilon$ in the following way;

Find $u \in H^1_0(B)$ such that

$$(P)''_\varepsilon \qquad (10) \quad -\operatorname{div} \sigma^e(u) + u(1-\chi) - \tilde{g} \cdot (\nabla \chi \cdot \tilde{n}) = \tilde{f}$$

$$\text{in B} \quad (\text{in } H^{-1}(B)).$$

Here $f = \tilde{f}$ (in Ω_0), 0 (in Ω_1). Also, $(\nabla \chi \cdot \tilde{n})$ is a well-defined distribution supported by Γ. Actually we have

$$(11) \quad \langle (\nabla \chi \cdot \tilde{n}) , \phi \rangle = \int_\Gamma \phi \, d\Gamma \quad \text{for any } \phi \in \mathcal{D}(B) \quad (\text{ see (25) }).$$

Let us prove the equivalence relation.

First, we show $(P)_\varepsilon \Longrightarrow (P)'_\varepsilon$.

By use of Green's theorem, we have from (3) and (4),

$$(12)\quad \nu a_0(u_0, v) + (u_0, v)_0 = \nu \int_\Gamma \frac{\partial u_0}{\partial n} v d\Gamma + (f, v)_0,$$

$$(13)\quad \varepsilon^{2\alpha} \cdot a_1(u_1, v) = - \varepsilon^{2\alpha} \int_\Gamma \frac{\partial u_1}{\partial n} v d\Gamma ,$$

By adding (12) and (13) and using (6), we have

$$(14)\quad \nu a_0(u_0, v) + \varepsilon^{2\alpha} a_1(u_1, v) + (u_0, v)_0 = \int_\Gamma g v d\Gamma + (f, v)_0$$

$$\text{for any } v \in H^1_0(B).$$

Next, we show $(P)'_\varepsilon \Longrightarrow (P)''_\varepsilon$.

We recall

$$(15)\quad \chi = \begin{cases} 1 & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_0. \end{cases}$$

(a) For $u \in H_0^1(B)$, we have

$$\nu a_0(u, v) + \varepsilon^{2\alpha} a_1(u, v) \tag{16}$$

$$= \int_B \{ \nu (1-\chi) \nabla u + \varepsilon^{2\alpha} \chi \nabla u \} \cdot \nabla v \, dx$$

$$= {}_{H^{-1}(B)}\left\langle -\operatorname{div}\{ \nu (1-\chi) \nabla u + \varepsilon^{2\alpha} \chi \nabla u \}, v \right\rangle_{H_0^1(B)}$$

for any $v \in H_0^1(B)$.

(b) For $g \in H^{1/2}(\Gamma)$, we have

$$\int_\Gamma g v \, d\Gamma = {}_{H^{-1}(B)}\left\langle \tilde{g} (\nabla \chi \cdot \tilde{n}), v \right\rangle_{H_0^1(B)} \tag{17}$$

for any $v \in H_0^1(B)$,

where $\tilde{g} \in H^1(B)$ and $\tilde{n}$ are the extensions of g and n into B.

In fact, let $\phi \in H^1(B)$. Then

(i) (18) $\phi \chi, \dfrac{\partial \phi}{\partial x_j} \cdot \chi \in L^2(B)$.

Define $T \equiv \phi \cdot \dfrac{\partial \chi}{\partial x_i} = \dfrac{\partial}{\partial x_j} (\phi \chi) - \dfrac{\partial \phi}{\partial x_j} \cdot \chi$

(ii) (19) $T \in \mathcal{D}'(B)$ $(j = 1, 2)$.

Moreover

(iii) (20) $T \in H^{-1}(B)$.

In fact, let $\phi \in \mathcal{D}(B)$. Then we have

$$(21) \quad \langle T, \phi \rangle = \langle \frac{\partial}{\partial x_j}(\phi \chi) - \frac{\partial \phi}{\partial x_j} \cdot \chi, \phi \rangle$$

$$= - \langle \phi \chi, \frac{\partial \phi}{\partial x_j} \rangle - \langle \frac{\partial \phi}{\partial x_j} \cdot \chi, \phi \rangle$$

from which follows

$$(22) \quad | \langle T, \phi \rangle | \leq \| \phi \|_{L^2} \cdot \| \frac{\partial \phi}{\partial x_j} \|_{L^2} + \| \frac{\partial \phi}{\partial x_j} \|_{L^2} \cdot \| \phi \|_{L^2}$$

$$\leq \| \phi \|_{H^1} \cdot \| \phi \|_{H^1} .$$

$\mathcal{D}(B)$ is dense in $H^1_0(B)$ and T is continuous from $\mathcal{D}(B)$ with H^1-norm. Therefore $T \in \mathcal{D}'(B)$ can be uniquely extended to $\tilde{T} \in H^{-1}(B)$ and again we denote $\tilde{T}$ by T.

(iv) Let Γ be smooth. Then

$$(23) \quad \langle T, \phi \rangle = \int_\Gamma \phi \phi n_j \, d\Gamma \qquad \text{for } \phi \in H^1_0(B).$$

Where n_j is the jth component of n.
In fact,

$$(24) \quad \langle T, \phi \rangle = - \langle \chi, \frac{\partial(\phi \phi)}{\partial x_j} \rangle$$

$$= - \int_{\Omega_1} \frac{\partial(\phi \phi)}{\partial x_j} dx = \int_\Gamma \phi \phi \, n_j \, d\Gamma .$$

(v) Let Γ be smooth and $\tilde{n}$ be a smooth extended vector field in B satisfying $\tilde{n}|_{\Gamma} = n$. Then

$$(25) \quad \langle \phi(\nabla\chi\cdot\tilde{n}),\ \phi\rangle = \int_{\Gamma} \phi\,\phi\, d\Gamma \quad \text{for } \phi \in H^1_0(B).$$

In fact, we may note that

$$\phi(\nabla\chi\cdot\tilde{n}) = \sum_{j=1}^{2} \phi\,\tilde{n}_j \cdot \frac{\partial\chi}{\partial x_j} .$$

By use of (a) and (b), we have distribution equation $(P_\varepsilon)''$.

Finally we show $(P_\varepsilon)'' \Longrightarrow (P)_\varepsilon$.

It is obvious that the solution of $(P_\varepsilon)''$ satisfies (14).

If we take $v \in \mathcal{D}(\Omega_0)$ in (14), then we have

$$(26) \quad -\nu\triangle u_0 + u_0 = f \qquad \text{in } \Omega_0, \quad (\text{in } \mathcal{D}'(\Omega_0)).$$

Because $\mathcal{D}(\Omega_0)$ is dense in $H^1(\Omega_0)$ and $u_0 \in H^1(\Omega_0)$, (26) holds in $H^{-1}(\Omega_0)$.

Similarly, taking $v \in \mathcal{D}(\Omega_1)$ in (10), we have

$$(27) \quad -\varepsilon^{2\alpha} \cdot \Delta u_1 = 0 \quad \text{in } \Omega_1, \quad (\text{in } H^{-1}(\Omega_1)).$$

Furthermore take $v \in \mathcal{D}(B)$ in (12), then by using (26), (27) and Green's theorem and repeating the density argument, we have

$$(28) \quad \nu \cdot \frac{\partial u_0}{\partial \nu} - \varepsilon^{2\alpha} \cdot \frac{\partial u_1}{\partial n} = g \quad \text{on } \Gamma, \quad (\text{in } H^{-1/2}(\Gamma_0)).$$

Now let us prove the convergence of u_ε. It is a known result that $(P)'_\varepsilon$ has a unique solution u_ε in $H^1_0(B)$ and u_ε is estimated as

$$(29) \quad \| u_\varepsilon \|_{H^1(\Omega_0)} \leq C_1 (\| f \|_{L^2(\Omega_0)} + \| g \|_{H^{-1/2}(\Gamma)}),$$

$$(30) \quad \varepsilon^{2\alpha} \| u_\varepsilon \|^2_{H^1(\Omega_1)} \leq C_2 ,$$

where C_1 and C_2 are positive constants [1].

Let us make $\varepsilon \to 0$. Then there exists a weak limit $u^0 \in H^1_0(\Omega\ )$; Namely

$$u_\varepsilon \to u^0 \quad \text{weakly in } H^1(\Omega_0) \text{ as } \varepsilon \to 0 \text{ along a suitable sequence.}$$

Since there holds

$$(31)\qquad \varepsilon^{2\alpha}\cdot|a_1(u_\varepsilon, v)| = \varepsilon^{\alpha}\cdot|a_1(\varepsilon^{\alpha}u_\varepsilon, v)|$$

$$< \varepsilon^{\alpha}\cdot(a_1(\varepsilon^{\alpha}u_\varepsilon, \varepsilon^{\alpha}u_\varepsilon))^{1/2}\cdot\|v\|_{H^1(\Omega_1)} \to 0 \qquad (\varepsilon \to 0),$$

we have, taking the limit in $(P)'_\varepsilon$,

$$(32)\qquad \nu\cdot a_0(u^0, v) + (u^0, v)_0 + \int_\Gamma g v\, d\Gamma = (f, v)_0$$

for any $v \in H^1_0(B)$, which is the variational formulation of (P) [26].

Here we note that whole of u_ε converges to u^0 weakly since the solution u^0 of (P) is unique.

2.2

Now we proceed to problems for vector valued function, again, with the Neumann boundary condition. As an example, we deal with a stationary Stokes-type problem defined in Ω_0, which has smooth boundary Γ.

Find $u \in H^1(\Omega_0)$ and $p \in L^2(\Omega_0)$ such that

(S)

(33) $-\nu \triangle u + \nabla p + u = f \quad \text{in } \Omega_0,$

(34) $\operatorname{div} u = 0 \quad \text{in } \Omega_0,$

(35) $\sigma_n(u) = g \quad \text{on } \Gamma,$

(36) $\sigma_t(u) = h \quad \text{on } \Gamma.$

Here $u = (u_1(x_1, x_2), u_2(x_1, x_2))$ is the velocity of the fluid, $p = p(x_1, x_2)$ is the pressure, $f = (f_1, f_2)$ is the external force,

$\sigma_n(u) = -p + \nu e_{ij}(u)n_i n_j$ is the normal component of the stress vector $[\sigma_{ij}(u,p)n_j]$ ($\sigma_{ij}(u,p) = -p\delta_{ij} + \nu e_{ij}(u)$, $e_{ij}(u) = \dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}$) on Γ, and $\sigma_t(u) = \nu\, e_{ij}(u)\, n_i t_j$ is the tangential component ($t=(t_1, t_2)$ is the unit tangential vector to Γ), and g, h are given smooth functions defined on Γ and ν is the coefficient of viscosity [23], [32], [38].

An approximate problem $(S)_\varepsilon$ for (S) by use of the fictitious domain method is the following:

$$\text{Find } u = u_0 \ (\text{in } \Omega_0),\ u_1 \ (\text{in } \Omega_1) \ \text{ and}$$

$$p = p_0 \ (\text{in } \Omega_0),\ 0 \ (\text{in } \Omega_1) \ \text{ such that}$$

$$(S)_\varepsilon \quad \begin{cases} (37) & -\nu\triangle u_0 + \nabla p_0 + u_0 = f & \text{in } \Omega_0, \\ (38) & \operatorname{div} u_0 = 0 & \text{in } \Omega_0, \\ (39) & -\varepsilon^{2\alpha}(\triangle u_1 + \nabla(\operatorname{div} u_1)) = 0 & \text{in } \Omega_1, \\ (40) & u_0 = u_1 & \text{on } \Gamma, \\ (41) & \sigma_n(u_0) = \varepsilon^{2\alpha}\cdot e_{ij}(u_1)\, n_i n_j + g & \text{on } \Gamma, \\ (42) & \sigma_t(u_0) = \varepsilon^{2\alpha}\cdot e_{ij}(u_1)\, n_i t_j + h & \text{on } \Gamma, \\ (43) & u_1 = 0 & \text{on } \partial B. \end{cases}$$

Variational formulation of $(S)_\varepsilon$ is as follows;

$$\text{Find } u \in (H_0^1(B))^2 \ \text{ and } p \in L^2(B) \ \text{ such that}$$

$$(S)'_\varepsilon \quad \begin{cases} (44) & \nu a_0(u, v) + \varepsilon^{2\alpha}\cdot a_1(u, v) + ((u, v))_0 - (p_0, \operatorname{div} v)_0 \\ & \displaystyle + \int_\Gamma g v_n \, d\Gamma + \int_\Gamma h v_t \, d\Gamma = (f, v)_0 \quad \text{for any } v \in (H_0^1(B))^2), \\ (45) & (q, \operatorname{div} u)_0 = 0, \quad \text{for any } q \in L^2(\Omega_0)). \end{cases}$$

Here $a_k(u, v) = \frac{1}{2} \sum_{i,j=1}^{2} \int_{\Omega_k} e_{ij}(u)\, e_{ij}(v)\, dx \qquad (k = 0, 1)$,

where $e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$,

and $((u, v))_0 = \sum_{i=1}^{2} \int_{\Omega_0} u_i \cdot v_i\, dx$, $(p_0, \operatorname{div} v)_0 = \int_{\Omega_0} p_0 \operatorname{div} v\, dx$.

By using $K_\sigma = \{ v \in (H_0^1(B))^2 \mid \operatorname{div} v = 0 \text{ in } \Omega_0 \}$,

we can reform $(S)'_\varepsilon$ as follows.

Find $u \in K_\sigma$ such that

$(S)'_{\varepsilon,\sigma}$ (46) $\nu a_0(u, v) + \varepsilon^{2\alpha} \cdot a_1(u, v) + ((u, v))_0 + \int_\Gamma g v_n\, d\Gamma$

$+ \int_\Gamma h v_t\, d\Gamma = ((f, v))_0$ for any $(v \in K_\sigma)$.

By virtue of a well known argument[38] it is clear that $(S)'_{\varepsilon,\sigma}$ has a unique solution u_ε in K_σ and u_ε is estimated as

(47) $\| u_\varepsilon \|_H \leq C_1(\nu) \cdot (\| f \|_{(L^2(\Omega_0))^2} + \| g \|_{H^{-1/2}(\Gamma)} + \| h \|_{H^{-1/2}(\Gamma)})$,

(48) $\varepsilon^{2\alpha} \cdot a_1(u_\varepsilon, u_\varepsilon) \leq C_2$.

Here $H = (H^1(\Omega_0))^2$.

Let $\varepsilon \to 0$, then there exists u^0 in H such that

(49) $$u_\varepsilon \to u^0 \quad \text{in } H.$$

Since there holds

(50) $$\varepsilon^{2\alpha} \cdot |a_1(u_\varepsilon, v)| \leq \varepsilon^\alpha (a_1(\varepsilon^\alpha u_\varepsilon, \varepsilon^\alpha u_\varepsilon))^{1/2} \cdot \|v\| \to 0 \ (\varepsilon \to 0),$$

we have from $(S)'_{\varepsilon,\sigma}$,

(51) $$\nu a_0(u^0, v) + ((u^0, v)) + \int_\Gamma g v_n \, d\Gamma + \int_\Gamma h v_t \, d\Gamma = (f, v)_0$$

for any $v \in K_\sigma$, from which (S) follows for a suitable choice of the additive constant in p.

Let us introduce the following extended flux ;

(52) $$\sigma^e_{ij}(u) = \{-p \cdot \delta_{ij} + \nu \cdot e_{ij}(u_0)\}(1 - \chi) + \varepsilon^{2\alpha} \cdot e_{ij}(u_1) \cdot \chi.$$

Define

(53) $$\nabla \chi = \left(\frac{\partial \chi}{\partial x_1}, \frac{\partial \chi}{\partial x_2}\right),$$

and

(54) $$\Sigma \chi = \left(\frac{\partial \chi}{\partial x_2}, -\frac{\partial \chi}{\partial x_1}\right).$$

Then the distribution equations for $(S)'_\varepsilon$ can be written as

$$(55)\quad -\frac{\partial}{\partial x_i}\sigma^e_{ij}(u) - \tilde{g}\cdot(\nabla\chi)_j - \tilde{h}\cdot(\Sigma\chi)_j + u\,(1-\chi)$$

$$(S)''_\varepsilon \qquad = f_j(1-\chi) \qquad \text{in } B\ (j = 1, 2) \quad (\text{in } H^{-1}(B)),$$

$$(56)\quad \operatorname{div} u\cdot(1-\chi) = 0 \qquad \text{in } B,$$

$$(57)\quad u = 0 \qquad \text{on } \partial B.$$

Concerning the equivalence among $(S)_\varepsilon$, $(S)'_\varepsilon$ and $(S)''_\varepsilon$, we can verify as in 2.1. In fact, the part of the verification of

$(S)_\varepsilon \Rightarrow (S)'_\varepsilon$ and $(S)'_\varepsilon \Rightarrow (S)''_\varepsilon$ is quite parallel and can be ommited. The part for $(S)''_\varepsilon \Rightarrow (S)_\varepsilon$ is briefly presented below.

We have $(S)'_\varepsilon$ from (55). Here we should note that crucial use has been made of the following relations ;

$$(58)\quad \left\langle \tilde{g}\frac{\partial\chi}{\partial x_j},\ \phi_j\right\rangle = \int_B \tilde{g}\frac{\partial\chi}{\partial x_j}\phi_j\,dx = \int_\Gamma g\,n_j\,\phi_j\,d\Gamma = \int_\Gamma g\,\phi_n\,d\Gamma$$

and

$$(59)\quad \langle \tilde{h}(\Sigma\chi)_j, \phi_j \rangle = \int_B \tilde{h}(\Sigma\chi)_j \phi_j \, dx = \int_\Gamma h\, t_j \phi_j \, d\Gamma.$$

$$= \int_\Gamma h \phi_t \, d\Gamma, \quad \text{for} \quad \phi \in (H_0^1(B))^2 .$$

Then we have from $(S)'_\varepsilon$,

$$(60)\quad \langle -\nu\triangle u_0 + u_0 + \nabla p, \phi \rangle_{\Omega_0} = \langle f, \phi \rangle$$

$$(\text{ for all } \phi \in (\mathcal{D}(\Omega_0)^2) \quad \text{namely.}$$

$$(61)\quad -\nu\triangle u_0 + u_0 + \nabla p = f \qquad \text{in } \Omega_0 .$$

Similarly we have

$$(62)\quad -\varepsilon^{2\alpha}(\triangle u_1 + \nabla \operatorname{div} u_1) = 0 \qquad \text{in } \Omega_1 .$$

By use of (61) and (62), we have

$$(63)\quad \langle \sigma_n(u_0), \phi_n \rangle_\Gamma = \langle \varepsilon^{2\alpha} \cdot e_{ij}(u_1) n_i n_j + g, \phi_n \rangle_\Gamma ,$$

$$(\text{for all } \phi \in (\mathcal{D}(B)^2 \cap \{ \phi_t = 0 \text{ on } \Gamma \}),$$

namely,

$$(64)\quad \sigma_n(u_0) = \varepsilon^{2\alpha} \cdot e_{ij}(u_1) n_i n_j + g \qquad \text{on } \Gamma .$$

Similarly, we have

$$(65)\quad \sigma_t(u_0) = \varepsilon^{2\alpha} \cdot e_{ij}(u_1) n_i t_j + h \qquad \text{on } \Gamma .$$

To be rigorous, (61), (62) and (64), (65) should be regarded as equations in $H^{-1}(\Omega_0)$, $H^{-1}(\Omega_1)$ and equalities in $H^{-1/2}(\Gamma)$, respectively.

We can apply the result thus obtained for the Stokes-like problem to the stationary Navier-Stokes-like problem, provided that the Reinolds number is sufficiently small. In this case, we must be cautious in dealing with

$$(66)\quad b(u,\ u,\ v)_0 = \int_{\Omega_0} u_i \frac{\partial u_j}{\partial x_i} v_j\, dx.$$

If we assume that

$$(67)\quad \nu^2 > C\ (\|f\|_{(L^2(\Omega_0))^2} + \|g\|_{H^{-1/2}(\Gamma)} + \|h\|_{H^{-1/2}}),$$

then we have the same result as in 2.1 [38].

The distribution equations $(NS)''_\varepsilon$ for the stationary Navier-Stokes-like equation with the stress boundary conditions can be constructed by use of the fictitious domain method similarly.

Remark 1.

In order to make the argument be simpler, we consider the Stokes-like problem mentioned above. As a matter of course, we could deal with the Stokes problem defined in Ω_0, which has an inner boundary Γ_0 except Γ, and prescribed with the Dirichlet boundary condition on Γ_0.

Remark 2.

Navier-Stokes equations may be treated in the same Ω_0 as stated in remark 1 when the Reinolds number is sufficiently small.

3. Application to flow problems with unilateral boundary conditions.

3.1

As some examples of penetrating or leaking phenomena of the fluid through the boundary, we can refer to the flow of air through a butterfly net and of gas or liquid through a filter. On the other hand, we observe typical slipping phenomena against surface with the flow of molten polymers of Bingham fluid in a pipe and of blood in a vein [7], [14]. In order to deal with such phenomena mathematically, we propose a model which is described by Navier-Stokes equations coupled with unilateral boundary conditions of the so called frictional type [34]; The latter boundary conditions which are composed of the following two conditions, now read ;

$$(\mathrm{S}) \qquad (1) \quad |\sigma_k(u)| \leqq g_k \qquad \text{on } \Gamma, \qquad (k = n,\ t)$$

and

$$(2) \quad \begin{cases} |\sigma_k(u)| < g_k & u_k = 0, \qquad \text{on } \Gamma, \\[2ex] |\sigma_k(u)| = g_k & \begin{cases} u_k = 0 \ \text{ or } \ u_k \neq 0 \\ \sigma_k(u) \cdot u_k < 0, \end{cases} \qquad \text{on } \Gamma. \end{cases}$$

To this slip boundary condition allows the following three equivalent expressions ;

$$(3) \quad \begin{cases} |\sigma_k(u)| \leq g_k, & \text{on } \Gamma, \\ u_k(g_k - |\sigma_k(u)|) = 0, & \text{on } \Gamma, \\ \sigma_k(u) \cdot u_k \leq 0. & \text{on } \Gamma. \end{cases}$$

$$(4)\quad \begin{cases} |\sigma_k(u)| \leq g_k, & \text{on } \Gamma, \\ \sigma_k \cdot u_k + g_k |\sigma_k(u)| = 0, & \text{on } \Gamma, \end{cases}$$

$$(5)\qquad -\nu \frac{\partial u_k}{\partial n} = g_k \partial(|u_k|). \qquad \text{on } \Gamma.$$

Here, on the right-hand side, $\partial(|\cdot|)$ means the multi-valued sub-differential at the number $\cdot$ of R^1-to-R^1 function $|\cdot|$.

Γ is a smooth part of the boundary $\partial\Omega_0$ of a domain $\Omega_0 \in R^2$ occupied by the fluid which obeys Navier-Stokes equations. Actually, we consider possible leak and / or slipping on Γ.

Suffix t and n represent tangential and normal compornents of the velocity field u and the stress $[\sigma_{ij}(u)\, n_i]$. g_k is a positive function defined on Γ, which means the threshhold for the normal or tangential stress which controls the occurrence of leak and / or slip.

Here we focus our attention on an inlet-outlet flow problem defined in a rectangular domain Ω_0 bounded by $\Omega_0 = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_w \cup \Gamma$.

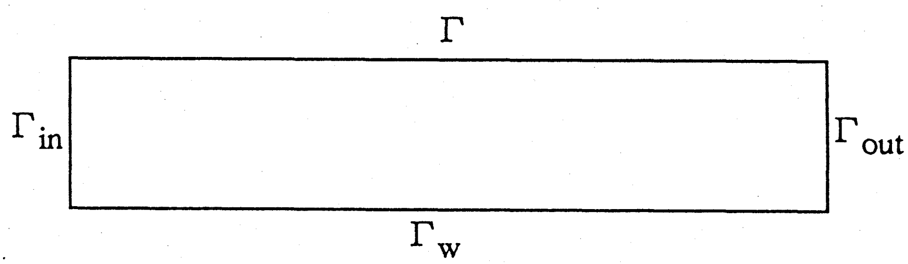

Figure 3.1

The model problem is described as follows ;

$$(P)_0 \quad \begin{cases} (6) & -\nu_0 \triangle u_0 + (u_0 \cdot \nabla) u_0 + \nabla p_0 = f_0 \quad \text{in } \Omega_0, \\ (7) & \operatorname{div} u_0 = 0 \quad \text{in } \Omega_0, \\ (8) & u_0 = \beta \quad \text{on } \Gamma_{in}, \\ (9) & \sigma_n(u_0) = 0 \quad \text{on } \Gamma_{out}, \\ (10) & u_{0t} = 0 \quad \text{on } \Gamma_{out}, \\ (11) & u_0 = 0 \quad \text{on } \Gamma_w, \\ (12) & u_0 \text{ satisfies (S) on } \Gamma, \end{cases}$$

Where p_0 is the pressure and f_0 is the external force. We can check that $(P)_0$ is transformed into the variational inequality [21],[28]. As for mathematical formulation and some crucial results in this connection, we refer to our works previously presented in an oral or informal way; [10],[11],[12],[13].

Let

$$K_0 = \{v \in (H^1(\Omega_0))^2 \mid v=\beta \text{ on } \Gamma_{in},\ v_t=0 \text{ on } \Gamma_{out} \text{ and } v=0 \text{ on } \Gamma_w\},$$

$$J(v) = \sum_{k=n,t} \int_\Gamma g_k |v_k| \, d\Gamma,$$

$$((u,v))_0 = \sum_{j=1}^{2} (u_j, v_j)_0 .$$

Where β is a vector valued function defined on Γ and $(\cdot,\cdot)_0$ means a scalar product in $L^2(\Omega_0)$.

Find u_0 and p_0 such that

$(V.I)_0$ (13) $\nu_0 \cdot a_0(u_0, v - u_0) + (((u_0 \cdot \nabla)u_0, v - u_0))_0$

$$- (p_0, \mathrm{div}(v-u_0))_0 + J(v) - J(u_0) \geqslant ((f_0, v - u_0))_0$$

for any $v \in K_0$,

(14) $(q, \mathrm{div}\, u_0)_0 = 0$ for any $q \in L^2(\Omega_0)$.

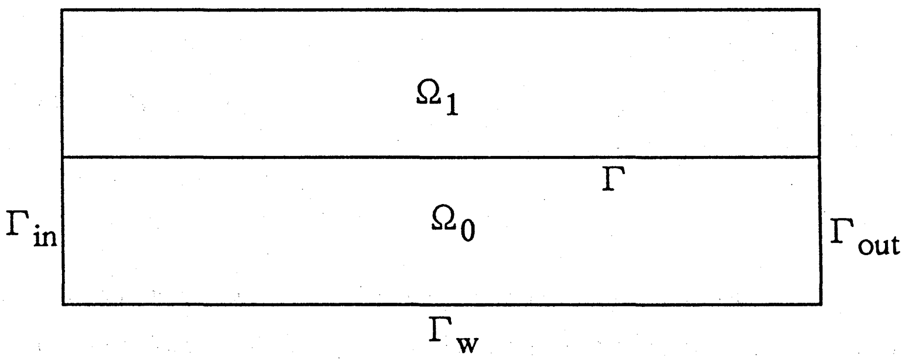

Figure 3.2

3.2

Furthermore we approximate $(V.I)_0$ by use of a singular-perturbed variational inquality defined in B by introducing a fictitious domain Ω_1, which is pasted to Ω_0 along their common boundary Γ. (see figure 3.2)

Let $K = \{v \in (H^1(B))^2 \mid v=\beta$ on Γ_{in}, $v_t=0$ on Γ_{out} and $v=0$ on $\Gamma_w \cup (\partial\Omega_1 \setminus \Gamma)\}$

Find $u^{\varepsilon} \in K$ and $p_0^{\varepsilon} \in L^2(\Omega\ )$ such that

$$(15) \quad \nu \cdot a_0(u^{\varepsilon}, v - u^{\varepsilon}) + \varepsilon^{2\alpha} \cdot a_1(u^{\varepsilon}, v - u^{\varepsilon})$$

$$(\mathrm{V.I})_{\varepsilon} \qquad + ((\,(u_0 \cdot \nabla) u_0,\ v - u_0))_0 - (p_0^{\varepsilon},\ \mathrm{div}\,(v - u^{\varepsilon}))_0$$

$$+ J(v) - J(u^{\varepsilon}) \geqslant ((f,\ v - u^{\varepsilon}))_0 \qquad \text{for } \forall\, v \in K,$$

$$(16) \quad (q,\ \mathrm{div}\, u^{\varepsilon})_0 = 0 \qquad \text{for } q \in L^2(\Omega_0).$$

This may be regarded as another example of the regularization from the outside[29]. The present singular perturbation defined in Ω_1 is given in the added fictitious domain Ω_1. Furthermore, we regularize $J(v)$ to obtain differentiable $J_{\varepsilon}(v)$ by use of $\phi_{\varepsilon}(\lambda) = \int_0^{\lambda} \tanh(\varepsilon^{-2\delta} s)ds$, ($\delta$ is a fixed positive constant) in the following way [14] :

$$(17) \quad J_{\varepsilon}(v) = \sum_{k=n,t} \int_{\Gamma} g_k \cdot \phi_{\varepsilon}(v_k)\, d\Gamma.$$

It is easily checked that the regularized singular-perturbed variational inequality is reduced to the following equations $(P)_{\varepsilon}$ with nonlinear transmission conditions on Γ ;

$(P)_\varepsilon$

(18) $-\nu_0 \triangle u_0^\varepsilon + (u_0^\varepsilon \cdot \nabla) u_0^\varepsilon + \nabla p_0^\varepsilon = f_0$ in Ω_0,

(19) $\mathrm{div}\, u_0^\varepsilon = 0$ in Ω_0,

(20) $-\varepsilon^{2\alpha} (\triangle u_1^\varepsilon + \nabla \mathrm{div}\, u_1^\varepsilon) = 0$ in Ω_1,

(21) $\sigma_n(u_0^\varepsilon) = \varepsilon^{2\alpha} \cdot e_{ij}(u_1^\varepsilon) n_i n_j + g_n \cdot \tanh(\varepsilon^{-2\delta} \cdot u_{on}^\varepsilon)$

on Γ,

(22) $\sigma_t(u_0^\varepsilon) = \varepsilon^{2\alpha} \cdot e_{ij}(u_1^\varepsilon) n_i t_j + g_t \cdot \tanh(\varepsilon^{-2\delta} \cdot u_{ot}^\varepsilon)$

on Γ,

(23) $u_0^\varepsilon = u_1^\varepsilon$ on Γ,

(24) $u_0^\varepsilon = \beta$ on Γ_{in},

(25) $u_{0t}^\varepsilon = 0$ on Γ_{out},

(26) $\sigma_n(u_0^\varepsilon) = 0$ on Γ_{out},

(27) $u_0^\varepsilon = 0$ on Γ_W,

(28) $u_1^\varepsilon = 0$ on $\partial\Omega_1 \setminus \Gamma$.

If we follow the distribution formulation mentioned in 2.2, then we have

Find $u^{\varepsilon} \in K$ and $P^{\varepsilon} \in L^{2}(\Omega_{0})$ such that

$$(29) \qquad -\frac{\partial}{\partial x_i} \sigma^{e}_{ij}(u^{\varepsilon}) + (u^{\varepsilon} \cdot \nabla) u^{\varepsilon} (1-\chi)$$

$$(P)'_{\varepsilon} \qquad + \tilde{g}_n \cdot \widetilde{\tanh(\varepsilon^{-2\delta} \cdot u^{\varepsilon}_{on})} (\nabla \chi)_j + \tilde{g}_t \cdot \widetilde{\tanh(\varepsilon^{-2\delta} \cdot u^{\varepsilon}_{ot})} (\Sigma \chi)_j$$

$$= f_j (1-\chi) \qquad \text{in } B, \quad (j=1,2)$$

$$(30) \qquad \operatorname{div} u^{\varepsilon} \cdot (1-\chi) = 0 \qquad \text{in } B.$$

and

u^{ε} satisfies the same boundary conditions on ∂B as in $(P)_{\varepsilon}$.

3.3 Numerical Results.

Hereafter, we report on the numerical solution of $(P)_{\varepsilon}$ solved by means of both finite difference approximation and MAC methods [2]. Let Ω_i $(i=0,1)$ be a rectangle whose size is 4×10. In this calculation, we chose $\nu = 10^{-2}$ and $\beta = (10, 0)$. The parameters h and g in Figers 3.3 ～3.7 mean $h = g_t$ and $g = g_n$.

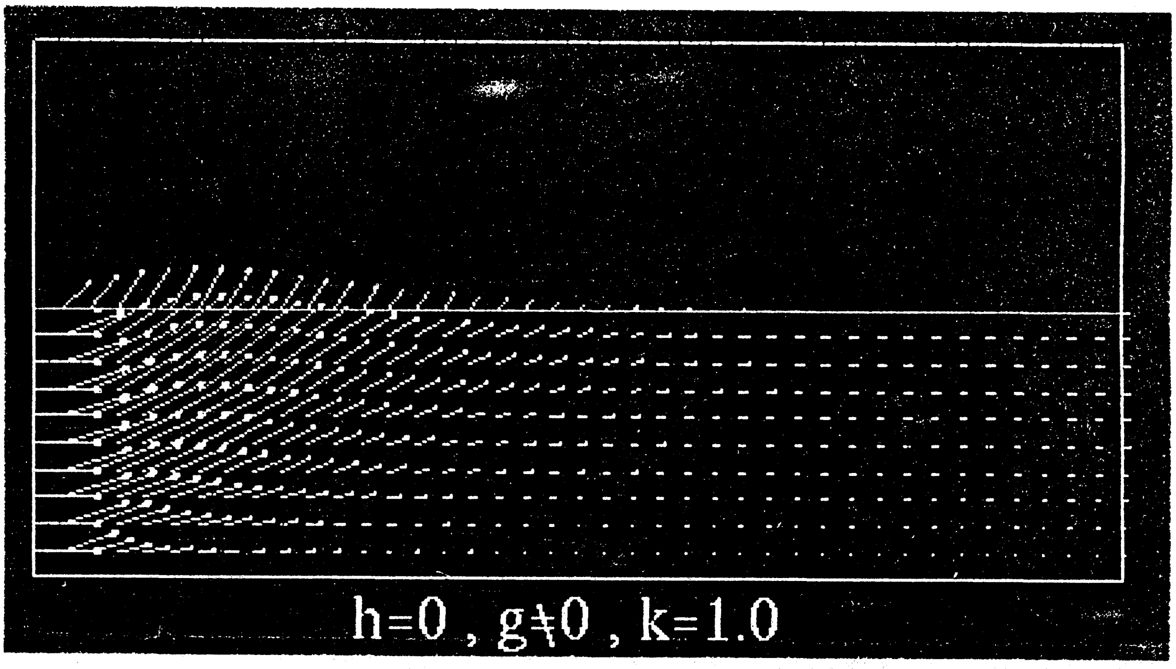

Fig. 3.3　 $k = g_n$

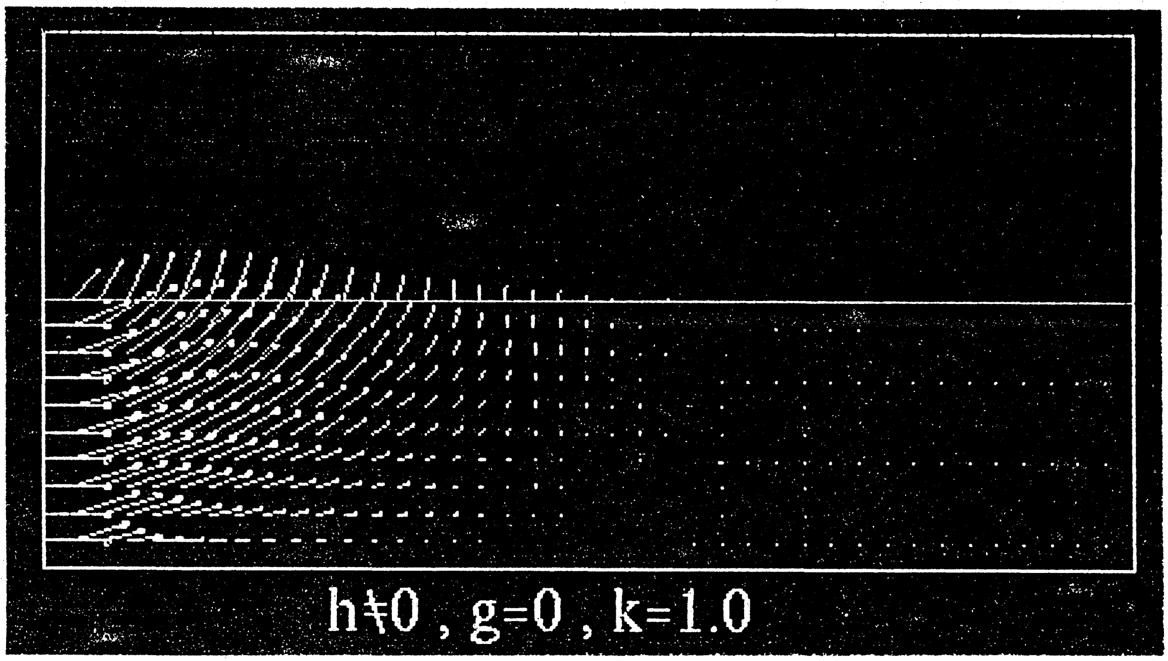

Fig. 3.4　 $k = g_t$

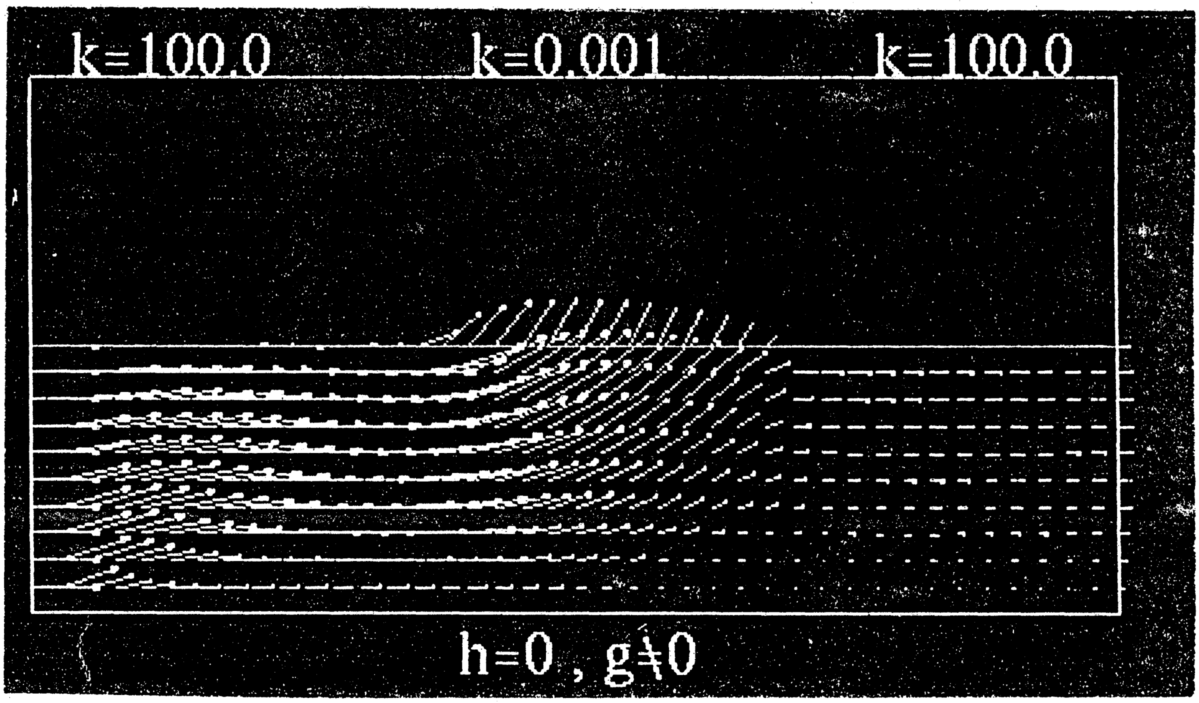

Fig. 3.5　k = g_n

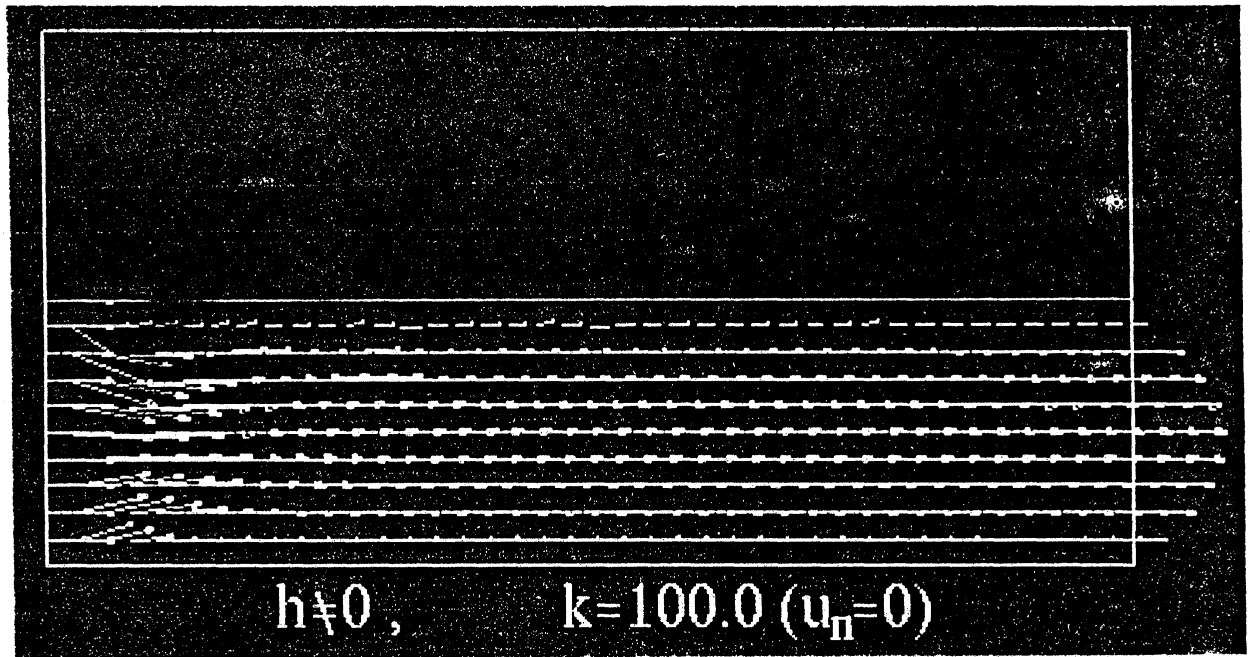

Fig. 3.6　u_n = 0 on Γ, k = g_t

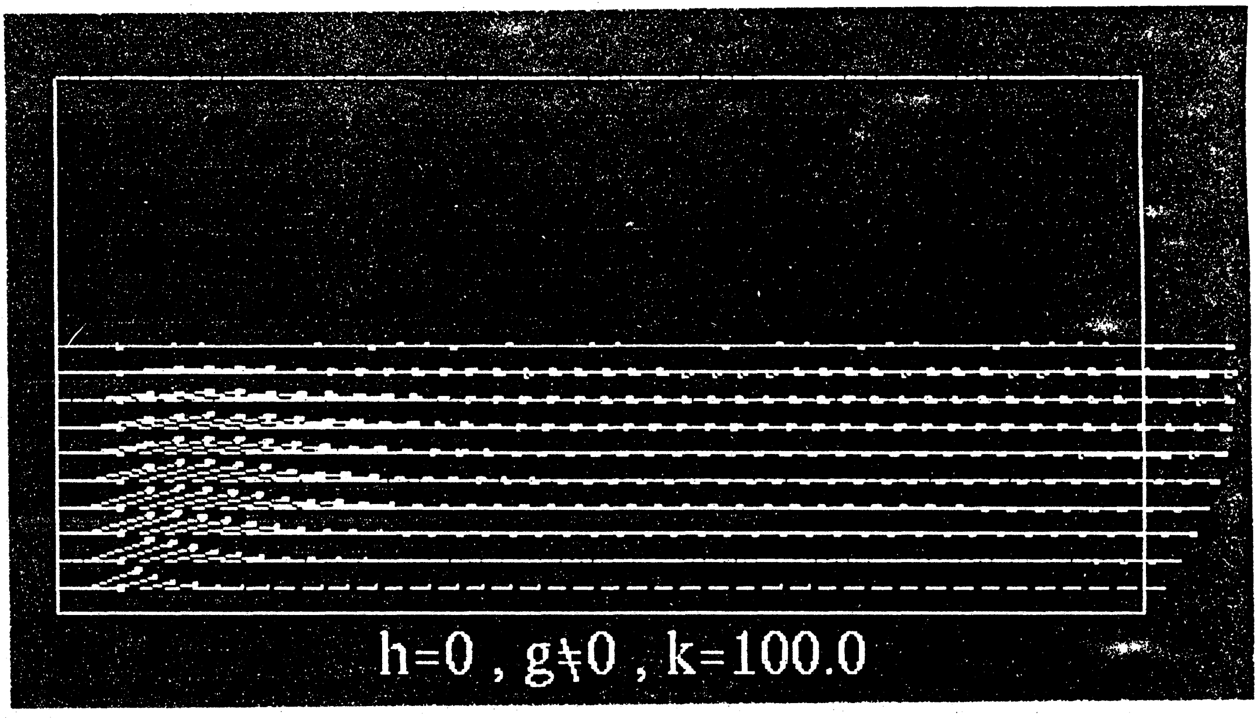

Fig. 3.7 $k = g_n$

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